PROOF PORTFOLIO: PROBLEM GROUP A

Directions. For each conjecture in the portfolio, determine whether the statement is true or false. If the statement is true, prove it. If not, provide a valid counterexample, as well as a slightly modified statement that is true, and prove the slightly modified statement.

If a biconditional statement is found to be false, you should clearly determine if one of the conditional statements within it is true; in that case, you should provide a proof of the conditional statement that holds. A counterexample for the statement that fails to hold suffices in this case (that is, you need not include an updated true statement and its proof for the one direction that is false).

Problems 1-3

Choose 3 of the following 7 problems to complete Group A, according to these guidelines: choose one of A1 and A2, one of A3 and A4, and one of A5, A6, and A7.

Conjecture A1.[1] If a is a Type 2 integer and b is a Type 1 integer, then $a^3 + 2b^2$ is a Type 2 integer.

Conjecture A2.[2] If n is an odd integer, then $16 \mid (n^4 + 4n^2 + 10)$.

For Problems A3 and A4, you will in some way use the Pythagorean Theorem, which we will accept to be true without proof: "For any right triangle, if its legs have length a and b and its hypotenuse length c, then $a^2 + b^2 = c^2$." In addition, two additional definitions are relevant to the conjectures you choose from below: a *triangular triple* (p, q, r) is a triple of natural numbers $p < q < r$ such that $p^2 + q^2 = r^2$. For instance, $(3, 4, 5)$ is a triangle triple. An *equilateral triangle* is one for which all three sides have equal length.

Conjecture A3. If k is a natural number and $k \geq 2$, then $(2k, k^2 - 1, k^2 + 1)$ is a triangular triple.

Conjecture A4. Let a right triangle with legs of length a and b and hypotenuse c be given. If an equilateral triangle is constructed on each side of the given right triangle, then the area of the equilateral triangle on the hypotenuse is equal to the sum of the areas of the equilateral triangles constructed on the two legs of the right triangle.

Conjecture A5. For any integer n, $3 \mid (2n^2 + 1)$ if and only if $3 \nmid n$.

Conjecture A6. For any integer k, $k^2 + 2k + 3$ is even if and only if $4 \mid (k^2 - 2k - 7)$.

Conjecture A7. For any integer a, $2a^2 + 3a \equiv 2 \pmod 7$ if and only if $a \equiv 5 \pmod 7$.

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[1] Type 0, 1, and 2 integers are defined in Exercise 9 of Section 1.2 of our text.

[2] As we will learn early in the course, we use the notation "$16 \mid k$," and read "16 divides the integer k" provided that there exists an integer m such that $k = 16m$. Similar notation applies to other integers; e.g., "$3 \mid k$" means that "3 divides k."