Proof Portfolio: Problem Group C

Directions. For each conjecture in the portfolio, determine whether the statement is true or false. If the statement is true, prove it. If not, provide a valid counterexample, as well as a slightly modified statement that is true, and prove the slightly modified statement.

If a biconditional statement is found to be false, you should clearly determine if one of the conditional statements within it is true; in that case, you should provide a proof of the conditional statement that holds. A counterexample for the statement that fails to hold suffices in this case (that is, you need not include an updated true statement and its proof for the one direction that is false).

Problems 8-10

Choose 3 of the following 6 problems to complete Group C, according to these guidelines: choose one of C1-C2, and two of C3-C6.

Conjecture C1. The function $g : \mathbb{R}\backslash\{3\} \to \mathbb{R}\backslash\{2\}$ defined by $g(x) = \dfrac{2x+1}{x-3}$ is both one-to-one and onto.

Conjecture C2. The function $L : \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$ given by $L(x,y) = (7x+4y, 5x+3y)$ is one-to-one and maps onto $\mathbb{R}\times\mathbb{R}$.

(A proof or disproof of Conjecture C2 must be based directly on the definitions of "one-to-one" and "onto"; it is not acceptable to use a result from another class.)

Conjecture C3. Let $\mathcal{M}$ represent the set of all 2×2 matrices with real entries. Define the determinant function $\det : \mathcal{M} \to \mathcal{M}$ by the rule

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$$

If $\mathcal{S} = \{A \in \mathcal{M} | \det(A) = 1\}$, then $\mathcal{S}$ is closed under the operation of matrix multiplication.

(A proof or disproof of Conjecture C3 must be based directly on the definition of "determinant"; it is not acceptable to use a result from another class.)

Conjecture C4. Let f be a differentiable function such that $f(0) = 0$. The function f is odd[1] if and only if its derivative, f', is even.

Conjecture C5. If f and g are real-valued functions that are each continuous on the interval $[a,b]$, differentiable on (a,b), and that satisfy $f(a) = g(a)$ and $f(b) = g(b)$, then there exists a value c such that $a < c < b$ and $f'(c) = g'(c)$.[2]

Conjecture C6. For each $n \in \mathbb{Z}$, n is the sum of 8 consecutive integers if and only if $n \equiv 4 \pmod 8$.

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[1] Recall from precalculus that a function $f : \mathbb{R} \to \mathbb{R}$ is *even* if for every $x \in \mathbb{R}$, $f(-x) = f(x)$. Likewise, a function is *odd* if for every $x \in \mathbb{R}$, $f(-x) = -f(x)$.

[2] If you do not recall the statement of Rolle's Theorem from calculus I, please stop by my office and we'll discuss it.